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# Analysis of the Impact of Wireless Three-User Multiple Access Channel Coefficients Correlation on Outage Probability: A Copula-Based Approach*

Research Note

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**Abstract**: In wireless communication channels, the channel coefficients are dependent on each other. In this paper, we investigate the constructive or destructive effects of fading coefficients correlation of wireless multiple access channels (MAC) on the outage probability (OP) (one of the wireless communications performances). Our desired channel is a wireless Rayleigh fading MAC with three-users, independent sources, and interdependent fading coefficients; we use the Copula theory to model the dependence structure. We obtain a closed-form expression for the channel outage probability using an important family of Copulas named the Farlie-Gumbel-Morgenstern (FGM) Copula. The results show that the negative dependence structure of the channel coefficients has a constructive effect on the outage probability; in other words, it reduces the outage probability values compared to the independent case. in contrast, the positive dependency has a destructive effect on the outage probability. The efficiency of the analytical results is illustrated numerically, and the numerical illustrations confirm our theoretical results.

**Keywords:** multiple access channel, correlated Rayleigh fading, Copula theory, Farlie-Gumbel-Morgenstern, outage probability

## 1. Introduction

In the last few decades, wireless communication has become an indispensable part of life and the growth of wireless networks continues. Considering the ever-increasing progress of wireless communication systems, as well as from practical and theoretical viewpoints, it is important to analyze the performance of communication systems using the results of information theory. In addition, the coefficients of communication channels in real wireless systems are dependent, and this dependency affects communication performances.

Copulas are known as a useful and powerful tool for modeling the dependence between random variables, hence these functions play an important role in various fields such as finance, insurance, hydrology, image processing statistics, machine learning, and various engineering applications [1-6]. Recently the use of Copula functions in the field of wireless communication is increasing.

An important family of Copulas is the FGM Copulas, first studied by Eyraud, Farlie, Gumble, and Morgenstern [7-10]. Some features of this family of Copulas have been studied in the literature [11-13]. FGM Copula is one of the fastest Copulas for statistical data processing and the simplest for calculating joint distributions [12]. Due to the simple form of this Copula, and the coverage of positive and negative values of its dependency parameter, the FGM Copula is very suitable for the analysis of wireless channels with dependent coefficients and provides a deep understanding of the effects of correlation on the performance metrics of multi-user communication systems.

The outage probability is an important performance metric of wireless communication, which has been studied in the literature [14-16]. In [14], assuming the correlation of the channel coefficients, a general closed-form expression of the outage probability for any arbitrary fading distribution has been obtained. Also, for the Rayleigh fading channel in two correlated and independent states, closed-form expressions for the outage probability have been obtained. In [15], for the doubly dirty fading MAC with non-causally known side information at transmitters, using the Copula theory, closed-form expressions for the outage probability and the coverage region are obtained. In [16], the authors have investigated the destructive and constructive effects of coefficients dependence of wireless three-user MAC on the outage probability.

Considering the importance of wireless MAC in communication issues, in this paper, these channels have been studied and the outage probability under the influence of the correlation of channel coefficients has been evaluated. For this purpose, first, using the three-dimensional FGM Copula function, a closed-form expression for the outage probability of wireless fading MAC with three transmitters has been obtained, Then, according to this expression, the

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influence of the dependence of the channel coefficients on the outage probability has been evaluated. Considering that the FGM Copula includes both positive and negative dependencies, therefore, by changing the dependency parameter of this Copula within the allowed range, the effect of both types of dependencies on the outage probability can be investigated.

The structure of this paper is as follows: channel model and a brief review of Copula theory are described in sections 2. The outage probability is obtained in Section 3. Numerical results are in section 4 and the paper is concluded in section 5.

## 2. Channel and Basic Definitions

### *2.1. Channel*

Our desired channel in this paper is a three-transmitter wireless multiple access channel with independent sources and dependent Rayleigh fading coefficients (Figure 1). The received signal is

$$Y = h_{1D}X_1 + h_{2D}X_2 + h_{3D}X_3 + Z \tag{1}$$

$X_1$, $X_2$, and $X_3$ are the signals sent by the first, second, and third transmitters, respectively.

$Z$ is independent identically distributed (iid) Additive White Gaussian Noise (AWGN) with zero mean and variance $N$. $h_{iD}; i \in \{1,2,3\}$ are the fading coefficients of the communication channel.

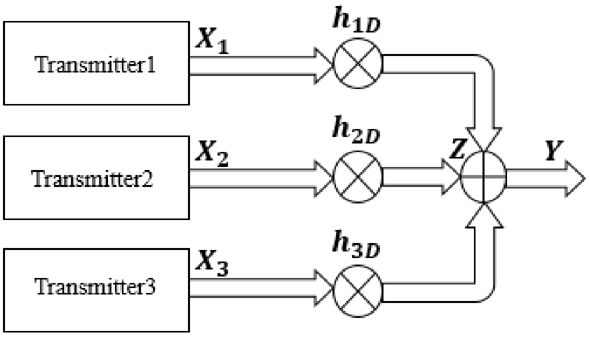

Figure 1. A three-user wireless Rayleigh fading MAC

We can extend the capacity region of two-user MAC with independent sources, determined by Ahlswede and Liao [17,18], to three-user MAC with independent sources.

The capacity region of a three-transmitter wireless MAC with block fading and coherent is equal to:

$$R_1 \le \frac{1}{2}\log_2\left(1 + \frac{P_1|h_{1D}|^2}{N}\right)$$

$$R_2 \le \frac{1}{2}\log_2\left(1 + \frac{P_2|h_{2D}|^2}{N}\right)$$

$$R_3 \le \frac{1}{2}\log_2\left(1 + \frac{P_3|h_{3D}|^2}{N}\right)$$

$$R_1 + R_2 \le \frac{1}{2}\log_2\left(1 + \frac{P_1|h_{1D}|^2 + P_2|h_{2D}|^2}{N}\right)$$

$$R_1 + R_3 \le \frac{1}{2}\log_2\left(1 + \frac{P_1|h_{1D}|^2 + P_3|h_{3D}|^2}{N}\right)$$

$$R_2 + R_3 \le \frac{1}{2}\log_2\left(1 + \frac{P_2|h_{2D}|^2 + P_3|h_{3D}|^2}{N}\right)$$

$$R_1 + R_2 + R_3 \le \frac{1}{2}\log_2\left(1 + \frac{P_1|h_{1D}|^2 + P_2|h_{2D}|^2 + P_3|h_{3D}|^2}{N}\right) \tag{2}$$

Where $R_1$, $R_2$ and $R_3$ are the desired transmission rates of the first, second, and third transmitters, respectively.

### *2.2. A Brief Review of Copula Theory*

What is a Copula? a brief review [1].

**Definition 1** A d-dimensional Copula, $C: [0,1]^d \rightarrow [0,1]$ is a function with the following properties:

1. $C(u_1, \dots, u_d) = 0; if\ any\ u_j = 0, j \in \{1, \dots, d\}$

2. $C\big(1, \dots, 1, u_j, 1, \dots, 1\big) = u_j$

3. $C$ is d-increasing on $[0,1]^{\mathrm{d}}$,that is:

$$\sum_{i_1=1}^{2} \dots \sum_{i_d=1}^{2} (-1)^{i_1+\dots+i_d} C\big(u_{1i_1}, \dots, u_{di_d}\big) \ge 0 \tag{3}$$

For all $0 \le u_{j1} \le u_{j2} \le 1$ and $j \in \{1, \dots, d\}$.

**Theorem 1** Assuming that $F$ is a d-dimensional cumulative distribution function (CDF) and $F_1, \dots, F_d$ are its margins, then there exists a Copula, $C$, such that:

$$F(x_1, \dots, x_d) = C\big(F_1(x_1), \dots, F_d(x_d)\big) \tag{4}$$

If $F_i$ is continuous for all $i = 1, \dots, d$, then $C$ is unique.

Using Sklar's theorem, the joint probability dens0ity function (PDF) corresponding to $F$ is as follows:

$$f(x_1, \dots, x_d) = f_1(x_1) \dots f_d(x_d) c\big(F_1(x_1), \dots, F_d(x_d)\big) \tag{5}$$

Where $f_i(x_i); i \in \{1, \dots, d\}$ are the marginal PDFs and $c$ is the Copula density function and is calculated as follows:

$$c(u_1, \dots, u_d) = \frac{\partial^d C(u_1, \dots, u_d)}{\partial u_1 \dots \partial u_d} \tag{6}$$

**Definition 2** A d-dimensional FGM Copula is defined as follows [11]:

$$C(u_1, \dots, u_d) = \left(\prod_{j=1}^{d} u_j\right)\left(1 + \sum_{k=2}^{d} \sum_{1 \le j_1 < \dots < j_k \le d} \theta_{j_1 \dots j_k} \bar{u}_{j1} \dots \bar{u}_{jk}\right) \tag{7}$$

Where $(u_1, \dots, u_d) \in [0,1]^d$ and $\overline{u}_j = 1 - u_j, j \in \{1, \dots, d\}$

## 3. Outage Probability

Outage probability is the probability that the channel capacity is less than a required threshold information rate, $R_{th}$.

**Theorem 2** The outage probability of a three-user wireless Rayleigh correlated fading MAC is:

$$P_{out} = 1 - \left[\frac{\bar{\lambda}_2 e^{-\frac{L}{\bar{\lambda}_2}}}{\left(\bar{\lambda}_2 - \bar{\lambda}_3\right)} + \boldsymbol{\theta}_{\boldsymbol{FGM}} \left( \frac{\boldsymbol{\bar{\lambda}_2 e^{-\frac{L}{\bar{\lambda}_2}} \left(1 + e^{-\frac{L}{\bar{\lambda}_2}}\right)}}{\boldsymbol{\left(\bar{\lambda}_2 - \bar{\lambda}_3\right)}} - \frac{\boldsymbol{2\bar{\lambda}_2 e^{-\frac{L}{\bar{\lambda}_2}}}}{\boldsymbol{\left(2\bar{\lambda}_2 - \bar{\lambda}_3\right)}} - \frac{\boldsymbol{\bar{\lambda}_2 e^{-\frac{2L}{\bar{\lambda}_2}}}}{\boldsymbol{\left(\bar{\lambda}_2 - 2\bar{\lambda}_3\right)}} \right)\right] \tag{8}$$

Proof: According to the definition given above for the outage probability, we have:

$$P_{out} = Pr(R_2 + R_3 \le R_{th}) \tag{9}$$

$$= 1 - Pr(R_2 + R_3 > R_{th}) \tag{10}$$

$$= 1 - P_{out}^c \tag{11}$$

Where $R_{th}$ represents the total required threshold information rates and $P_{out}^c$ is the complement of the outage probability. Now we calculate $P_{out}^c$.

Any of the inequalities in (2) can be used to calculate $P_{out}^c$. In this paper, we use the following inequality:

$$R_2 + R_3 \le \frac{1}{2}\log_2\left(1 + \frac{P_2|h_{2D}|^2 + P_3|h_{3D}|^2}{N}\right) \tag{12}$$

We have:

$$P_{out}^c = Pr\left(\frac{1}{2}\log_2\left(1 + \frac{P_2|h_{2D}|^2 + P_3|h_{3D}|^2}{N}\right) > R_{th}\right) \tag{13}$$

$$= Pr\left(\frac{P_2|h_{2D}|^2 + P_3|h_{3D}|^2}{N} > 2^{2R_{th}} - 1\right) \tag{14}$$

$$= Pr(\lambda_2 + \lambda_3 > L) \tag{15}$$

$$= \int_0^\infty \int_{L-\lambda_3}^\infty f(\lambda_2, \lambda_3) d\lambda_2 d\lambda_3 \tag{16}$$

Where $\lambda_2 = \frac{P_2|h_{2D}|^2}{N}$, $\lambda_3 = \frac{P_3|h_{3D}|^2}{N}$, $L = 2^{2R_{th}} - 1$ and $f(\lambda_2, \lambda_3)$ is the joint PDF of $\lambda_2$ and $\lambda_3$.

To calculate the double integral (16), we need to have the joint PDF of $\lambda_2$ and $\lambda_3$. Also, according to (5), to calculate $f(\lambda_2, \lambda_3)$, we must have the marginal PDFs, $f(\lambda_2)$ and $f(\lambda_3)$, and the Copula density function, $c(u_1, u_2)$.

As stated in subsection *A*, the channel coefficients, $h_{iD}; i \in \{1,2,3\}$, have a Rayleigh distribution and we know that the square of the Rayleigh distribution is an exponential distribution, so $|h_{iD}|^2; i \in \{1,2,3\}$ and consequently $\lambda_i; i \in \{1,2,3\}$ have an exponential distribution:

$$f(\lambda_i) = \frac{1}{\bar{\lambda}_i} exp\left(-\frac{\lambda_i}{\bar{\lambda}_i}\right); i \in \{1,2,3\} \tag{17}$$

$$F(\lambda_i) = 1 - exp\left(-\frac{\lambda_i}{\bar{\lambda}_i}\right); i \in \{1,2,3\} \tag{18}$$

Where $f(\lambda_i)$; $i \in \{1,2,3\}$ and $F(\lambda_i)$; $i \in \{1,2,3\}$ are PDFs and CDFs of $\lambda_i; i \in \{1,2,3\}$, respectively. $\bar{\lambda}_i; i \in \{1,2,3\}$ are average SNRs and are given as:

$$\bar{\lambda}_i = \frac{P_i \mathbb{E}[|h_i|^2]}{N}; i \in \{1,2,3\} \tag{19}$$

Considering $d = 2$ in (7), two-dimensional FGM Copula is obtained as:

$$C_{FGM}(u_1, u_2) = u_1 u_2 (1 + \theta_{FGM} \overline{u}_1 \overline{u}_2);\ \theta_{FGM} \in [-1,1] \tag{20}$$

Where $\overline{u}_1 = 1 - u_1$, $\overline{u}_2 = 1 - u_2$ and $\theta_{FGM}$ is the FGM Copula parameter.

Now, according to (6) and (20), the density function of the two-dimensional FGM Copula is:

$$c_{FGM}(u_1, u_2) = 1 + \theta_{FGM}(1 - 2u_1)(1 - 2u_2) \tag{21}$$

According to (5), (17) and (21), $f(\lambda_2, \lambda_3)$ is obtained as follows:

$$f(\lambda_2, \lambda_3) = \frac{e^{-\frac{\lambda_2}{\bar{\lambda}_2} - \frac{\lambda_3}{\bar{\lambda}_3}}}{\bar{\lambda}_2 \bar{\lambda}_3}\left[1 + \theta_{FGM}\left(1 - 2e^{-\frac{\lambda_2}{\bar{\lambda}_2}}\right)\left(1 - 2e^{-\frac{\lambda_3}{\bar{\lambda}_3}}\right)\right] \tag{22}$$

By putting (22) in (16), it is easy to calculate the double integral and the outage probability is obtained as (8) and the proof is complete.

## 4. Numerical Results

In this section, we investigate the effect of the correlation of channel coefficients on the outage probability.

Figure 2 and Fig. 3 show the outage probability versus the average SNR. As can be seen, as average SNR increases, the outage probability decreases, because the channel conditions improve.

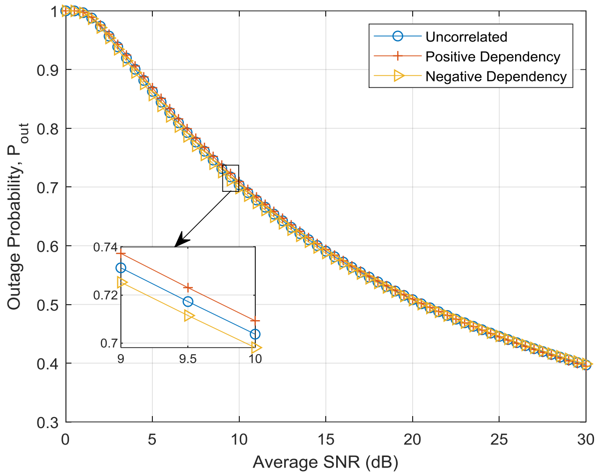

Figure 2. Outage probability versus average SNR, $\theta_{FGM} = 1/8$ (positive dependency), $\theta_{FGM} = 0$ (uncorrelated), and $\theta_{FGM} = -1/8$ (negative dependency)

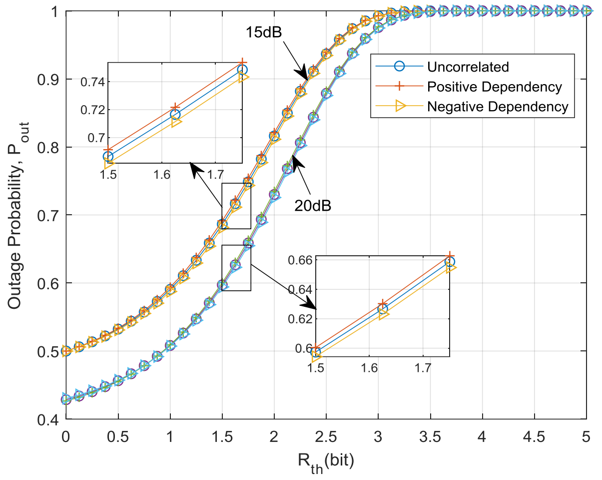

Figure 3. Outage probability versus $R_{th}$, $\theta_{FGM} = 1/8$ (positive dependency), $\theta_{FGM} = 0$ (uncorrelated), and $\theta_{FGM} = -1/8$ (negative dependency)

Also, we can see that the negative dependence structure ($\theta_{FGM} \epsilon [-1, 0)$) improves the outage probability performance compared to the uncorrelated state. In contrast, the positive dependence structure has a destructive effect on the outage probability performance.

Figure 3 and Figure 5 show the behavior of outage probability in terms of the total required threshold information. It can be seen that as $R_{th}$ increases, the outage probability also increases and finally tends to one, and this means that it is impossible to transfer information. According to these figures, it is clear that the outage probability under the influence of the negative dependence structure has lower values compared to the uncorrelated state.

The curves in Figure 3 are plotted for two different values of average SNR, and according to these two groups of curves, we find that for a fixed value of the total required threshold information rate, $R_{th}$, as the average SNR increases, the outage probability improves.

By comparing Figure 2 with Figure 4 and Figure 3 with Figure 5, we find that as the positive dependence increases, its destructive effect on the outage probability increases.

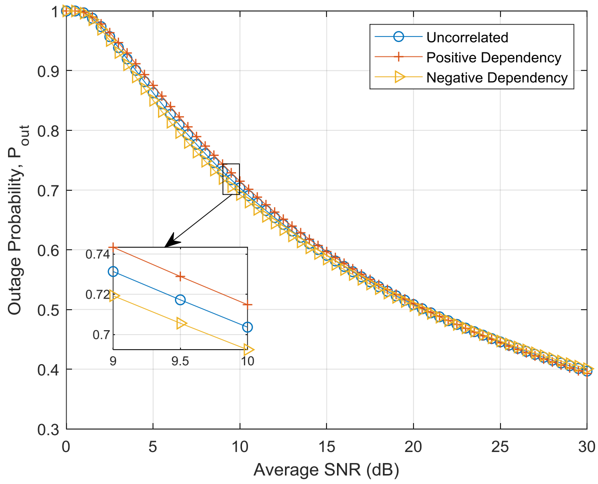

Figure 4. Outage probability versus average SNR, $\theta_{FGM} = 1/4$ (positive dependency), $\theta_{FGM} = 0$ (uncorrelated), and $\theta_{FGM} = -1/4$ (negative dependency)

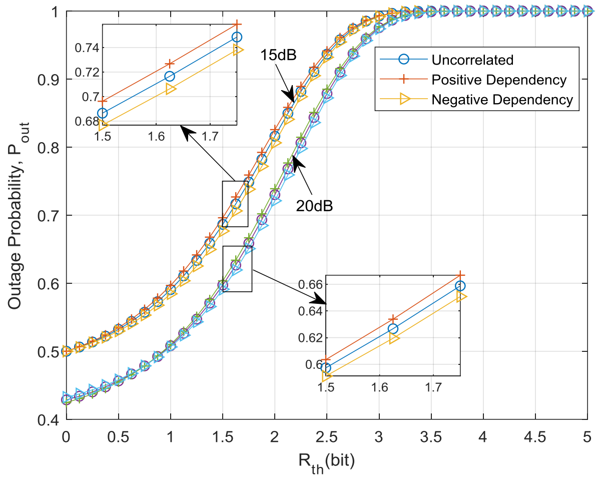

Figure 5. Outage probability versus $R_{th}$, $\theta_{FGM} = 1/4$ (positive dependency), $\theta_{FGM} = 0$ (uncorrelated), and $\theta_{FGM} = -1/4$ (negative dependency)

Similarly, as the negative dependence increases, its constructive effect on the outage probability increases.

## 5. Conclusion

In this paper, wireless three-user MAC with independent sources and Rayleigh fading was investigated. Using the FGM Copula, a closed form expression for the outage probability was obtained. Then we analyzed the impact of positive and negative dependencies on the outage probability performance. According to the obtained results, it is clear that negative dependence, compared to the independent state, reduces the outage probability, while positive dependency increases the outage probability compared to the non-dependent case.

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